Exercise 1.4 IMPORTANT TOPICS & EXAMPLES

1.10 Solving Problems Involving Exponential and Logarithmic Equations and Inequalities

Example 14: Solve the equation $4^{x-1}=8$.

Sol: The given equation is: $4^{x-1}=8$.

Domain: Since the expression 4^{x-1} is defined for every real number

Domain $=]-\infty,\infty[$.

Since $4=2^2$ and $8=2^3$, the above equation can be written as: $(2^2)^{x-1}=2^3$ or $2^{2x-2}=2^3$

Since the exponential function is one to one, we must have:

$$2x-2=3 \Rightarrow x=\frac{5}{2}$$

Solution Set: $\left\{\frac{5}{2}\right\}$

Example 15: Find the solution of $2^{2x}-3^{5x+3}=0$.

Sol: The given equation can be written as:

$2^{2x}=3^{5x+3}$ --------(1)

Domain: Since the expressions 2^{2x} and 3^{5x+3} are defined for every real number.

So, Domain $=]-\infty,\infty[$.

Applying natural logarithms on both sides of equation(1):

$\ln 2^{2x}=\ln 3^{5x+3}$

Applying power law of logarithms:

$2x\ln 2=(5x+3)\ln 3$

$\Rightarrow \quad 2x\ln 2=5x\ln 3+3\ln 3$

$\Rightarrow \quad (2\ln 2-5\ln 3)x=3\ln 3$

Applying power law of logarithms:

$(\ln 2^2-\ln 3^5)x=\ln 3^3$

$\Rightarrow \quad (\ln 4-\ln 243)x=\ln 27$

Applying quotient law of logarithms:

$$x\ln\left(\frac{4}{243}\right)=\ln 27$$

$$\Rightarrow x=\ln 27/\ln\left(\frac{4}{243}\right)=\log_{\frac{4}{243}}(27)$$

Solution Set: $\left\{\log_{\frac{4}{243}}(27)\right\}$

Example 16: Solve the equation: $\log x+\log(x-3)=1$

Sol: The given equation is:

$\log x+\log(x-3)=1$ -------(1)

Domain: Note the expressions $\log x$ and $\log(x-3)$ are defined only when:

$x>0$ and $x-3>0 \Rightarrow x>3$

Hence, Domain $=]3,\infty[$.

Applying product law of logarithms on the L.H.S of the given equation (1):

$\log[x(x-3)]=1$

Writing in exponential form:

$x(x-3)=10^1=10$

Factorizing: $x^2-3x-10=0$

$(x-5)(x+2)=0 \Rightarrow x=5$ or $x=-2$

Since $5\in]3,\infty[$ whereas, $-2\notin]3,\infty[$, the solutic set is $\{5\}$.

Example 17: Solve the equation: $\log_x(x^2-3x+2)=$

Sol: **Domain:**

- Base condition: $x>0,\ x\neq 1$
- Argument condition: $x^2-3x+2>0$

The given equation is: $\log_x(x^2-3x+2)=2$.

- Convert to exponential form: $x^2=x^2-3x+2$

$\Rightarrow \quad 3x-2=0 \Rightarrow 3x=2 \Rightarrow x=\frac{2}{3}$

Check domain:

- Base condition: $x=\frac{2}{3}>0$ and $x\neq 1$.
- Argument condition:

$$x^2-3x+2=\left(\frac{2}{3}\right)^2-3\left(\frac{2}{3}\right)+2=\frac{4}{9}-2+2=\frac{4}{9}>0$$

Hence, $x=\frac{2}{3}$ is a solution of the given equation

Solution set: $=\left\{\frac{2}{3}\right\}$

Example 18: Solve the inequality: $2^{x+1}>5^x$.

Sol: **Domain:** Note that, the expression 2^{x+1} and 5^x a defined for every real number.

The given inequality can be written as: $2.2^x>5$

Since, $2^x>0$ for every real number, we have:

$$2>\frac{5^x}{2^x} \Leftrightarrow 2>\left(\frac{5}{2}\right)^x$$

Since, the logarithmic function with base 10 is strictly increasing, we have:

$2 > \frac{5^x}{2^x} \Leftrightarrow \log 2 > \log\left(\frac{5}{2}\right)^x$

Applying power law of logarithms:

$\log 2 > x\log\left(\frac{5}{2}\right)$

Since, $\frac{5}{2} > 0 \Rightarrow \log\left(\frac{5}{2}\right) > 0,$

we have: $x < \frac{\log 2}{\log\left(\frac{5}{2}\right)} = \log_{\frac{5}{2}}(2) \approx 0.7565$

Hence, $2^{x+1} > 5^x \Rightarrow x < \log_{\frac{5}{2}}(2)$

Solution Set: $]-\infty, \log_{\frac{5}{2}}(2)[.$

Example 19: Solve the inequality:

$\log(2x-1) \geq \log(5-3x)$

Sol: **Domain:** Note that $\log(2x-1)$ and $\log(5-3x)$ are defined when:

$2x-1>0 \Rightarrow x > \frac{1}{2}$ *and* $5-3x>0 \Rightarrow x < \frac{5}{3}$

So, we have: $\frac{1}{2} < x < \frac{5}{3}$

The given inequality is: $\log(2x-1) \geq \log(5-3x)$

Since, the logarithmic function with base 10 is strictly increasing, we must have:

$\log(2x-1) \geq \log(5-3x) \Leftrightarrow 2x-1 \geq 5-3x$

Solve inequality:

$2x+3 \geq 5+1 \Leftrightarrow 5x \geq 6 \Leftrightarrow x \geq \frac{6}{5}$

Compatibility with domain: $\frac{6}{5} \leq x < \frac{5}{3}$

If follows that:

$\log(2x-1) \geq \log(5-3x)$ *iff* $\frac{6}{5} \leq x < \frac{5}{3}$

Solution set: $\left[\frac{6}{5}, \frac{5}{3}\right[$

Example 20: Solve the inequality:

$\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

Sol: **Domain:** for the expression $\log_{\frac{7}{9}}(x^2-3x+2)$ to be defined, we have:

$x^2-3x+2>0 \Rightarrow (x-1)(x-2)>0 \Rightarrow$ either $x<1$ and $x<2$ (which gives $x<1$) or $x>1$ and $x>2$ (which gives $x>2$). Hence $x<1$ *or* $x>2$

For the expression $\log_{\frac{7}{9}}(2x-1)$ to be defined,

we have: $2x-1>0 \Rightarrow x > \frac{1}{2}$

Combined domain: $\left]\frac{1}{2}, 1\right[\cup]2, \infty[$

The given inequality is:

$\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

Since, $0 < \frac{7}{9} < 1$, the logarithmic function with base $\frac{7}{9}$ is strictly decreasing. Thus,

We have:

$\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

$\Leftrightarrow x^2-3x+2 \leq 2x-1$

$\Leftrightarrow x^2-3x+2-2x+1 \leq 0$

$\Leftrightarrow x^2-5x+3 \leq 0$

Solving the quadratic equation $x^2-5x+3=0$

$x = \frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$

Since, $x^2-5x+3 \leq 0$, either $x \leq \frac{5-\sqrt{13}}{2}$ and $x \geq \frac{5+\sqrt{13}}{2}$ (which is impossible) or $x \geq \frac{5-\sqrt{13}}{2}$ and $x \leq \frac{5+\sqrt{13}}{2}$

(which gives $\frac{5-\sqrt{13}}{2} \leq x \leq \frac{5+\sqrt{13}}{2}$)

Compatibility with the domain:

$\left]\frac{1}{2}, 1\right[\cup]2, \infty[.$

If $x \in \left[\frac{1}{2}, 1\right[\cup]2, \infty[$ and $\frac{5-\sqrt{13}}{2} \leq x \leq \frac{5+\sqrt{13}}{2}$

then, we have:

$x \in \left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$

Hence, $\log_{\frac{7}{9}}(x^2-3x+2) \geq \log_{\frac{7}{9}}(2x-1)$

iff $x \in \left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$

Solution set: $\left[\frac{5-\sqrt{13}}{2}, 1\right[\cup \left]2, \frac{5+\sqrt{13}}{2}\right]$

Exercise 1.4 SOLUTION

1. Solve the following equations:

i) $\log(3x-2)=\log(x^2-5x+6)$

Sol. $\log(3x-2)=\log(x^2-5x+6)$

$3x-2=x^2-5x+6$

$x^2-5x+6-3x+2=0$

$1x^2-8x+8=0$

$a=1,\quad b=-8,\quad c=8$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{8\pm\sqrt{64-32}}{2}=\frac{8\pm\sqrt{32}}{2}$$

$$=\frac{8\pm\sqrt{16\times2}}{2}=\frac{8\pm4\sqrt{2}}{2}=\frac{2(4\pm2\sqrt{2})}{2}$$

$x=4\pm2\sqrt{2}$

$S.S=\{4-2\sqrt{2},4+2\sqrt{2}\}$

ii) $\ln(x^2-1)=\ln(3x-2)$

Sol. $\ln(x^2-1)=\ln(3x-2)$

$x^2-1=3x-2$

$x^2-3x-1+2=0$

$x^2-3x+1=0$

$$x=\frac{3\pm\sqrt{9-4}}{2}=\frac{3\pm\sqrt{5}}{2}$$

$$x_1=\frac{3-\sqrt{5}}{2}\approx0.382$$

$$x_2=\frac{3+\sqrt{5}}{2}\approx2.618$$

$x_1=0.382$ is invalid and x_2 satisfies $x>1$

$$\text{S.S.}=\left\{\frac{3+\sqrt{5}}{2}\right\}$$

iii) $\log(x)+\log(x^2-5x+7)=\log(3)$

Sol. $\log(x)+\log(x^2-5x+7)=\log(3)$

$\log[x(x^2-5x+7)]=\log(3)$

$x(x^2-5x+7)=3$

$x^3-5x^2+7x-3=0$

Using synthetic division $(x-1)$ is a factor.

$x^3-5x^2+7x-3=0$

$(x-1)(x^2-4x+3)=0$

$(x-1)(x-1)(x-3)=0$

$(x-1)^2(x-3)=0$

$x=1,3$

Hence $x=1,3$

Since $x_1=1$ and $x_2=3$ satisfies $x>0$

Hence S.S. $=\{1,3\}$

iv) $3^{(x+1)}-2\cdot9^x+9=0$

Sol. $3^{x+1}-2\cdot9^x+9=0$

$3^x\cdot3^1-2\cdot(3^2)^x+9=0$

$3\cdot3^x-2\cdot3^{2x}+9=0$

$3\cdot3^x-2\cdot(3^x)^2+9=0$

$-2(3^x)^2+3\cdot3^x+9=0\ \rightarrow(1)$

Put $u=3^x\ \rightarrow(2)$

$(1)\Rightarrow-2u^2+3u+9=0$

$2u^2-3u-9=0\qquad\therefore\ (\times\text{ by }-1)$

$2u^2-6u+3u-9=0$

$2u(u-3)+3(u-3)=0$

$(u-3)(2u+3)=0$

$$u=-\frac{3}{2},3$$

When $u=-\frac{3}{2},(2)\Rightarrow3^x=-\frac{3}{2}$

(Not possible because $3^x>0$)

When $u=3,(2)\Rightarrow3^x=3\Rightarrow3^x=3^1\Rightarrow x=1$

S.S. $=\{1\}$

v) $x\log_{\frac{1}{e}}4=\log_{\frac{1}{e}}(16^x-3)$

Sol. $x\log_{\frac{1}{e}}4=\log_{\frac{1}{e}}(16^x-3)$

$\log_{\frac{1}{e}}4^x=\log_{\frac{1}{e}}(16^x-3)$

$4^x=16^x-3$

$4^x=(4^2)^x-3$

$4^x=(4^x)^2-3\ \rightarrow(1)$

Put $4^x=u\ \rightarrow(2)$

$(1)\Rightarrow u=u^2-3$

$u^2-u-3=0$

$$u=\frac{1\pm\sqrt{1+12}}{2}=\frac{1\pm\sqrt{13}}{2}$$

When $u=\frac{1-\sqrt{13}}{2}$

$(2)\Rightarrow 4^x=\frac{1-\sqrt{13}}{2}$

(Not possible because $4^x>0$)

When $u=\frac{1+\sqrt{13}}{2}$

$(2)\Rightarrow 4^x=\frac{1+\sqrt{13}}{2}$

$(2^2)^x=\frac{1+\sqrt{13}}{2}\Rightarrow 2^{2x}=\frac{1+\sqrt{13}}{2}$

$2x=\log_2\left(\frac{1+\sqrt{13}}{2}\right)\Rightarrow x=\frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right)$

$S.S.=\left\{\frac{1}{2}\log_2\left(\frac{1+\sqrt{13}}{2}\right)\right\}$

vi) $2^x+2^{-x}=5$

Sol. $2^x+2^{-x}=5$

$(2^x)+(2^x)^{-1}=5 \quad \rightarrow (1)$

Put $2^x=u \quad \rightarrow (2)$

$(1)\Rightarrow u+u^{-1}=5$

$u+\frac{1}{u}=5 \qquad (\times \text{ by } u)$

$u^2+1=5u\Rightarrow u^2-5u+1=0$

$u=\frac{5\pm\sqrt{25-4}}{2}=\frac{5\pm\sqrt{21}}{2}$

When $u=\frac{5-\sqrt{21}}{2}$

$(2)\Rightarrow 2^x=\frac{5-\sqrt{21}}{2}$

$x=\log_2\left(\frac{5-\sqrt{21}}{2}\right)$

When $u=\frac{5+\sqrt{21}}{2}$

$(2)\Rightarrow 2^x=\frac{5+\sqrt{21}}{2}$

$x=\log_2\left(\frac{5+\sqrt{21}}{2}\right)$

$S.S.=\left\{\log_2\left(\frac{5-\sqrt{21}}{2}\right),\log_2\left(\frac{5+\sqrt{21}}{2}\right)\right\}$

2. Solve the following inequalities (if its solution exists)

i) $2^x+2^{1-x}\le 3$

Sol. $2x+2^1\cdot 2^{-x}\le 3$

$2^x+\frac{2}{2^x}\le 3$

$(2^x)^2+2\le 3\cdot 2^x \qquad (\times \text{ by } 2^x)$

$(2^x)^2-3\cdot 2^x+2\le 0 \quad \rightarrow (1)$

Put $2^x=u \quad \rightarrow (2)$

$(1)\Rightarrow u^2-3u+2\le 0$

$u^2-3u+2=0$

$(u-2)(u-1)=0$

$u=1,2$

Since $1\le u\le 2$ (because $u^2-3u+2\le 0$)

Put $u=2^x$ from (2)

$1\le 2^x\le 2$

$\log_2(1)\le x\le \log_2(2)$

$0\le x\le 1$

$S=[0,1]$

ii) $2^x\ge \log_2(16)$

Sol. $2^x\ge \log_2(2^4)$

$2^x\ge 4\cdot\log_2(2)$

$2^x\ge 4\cdot 1$

$2^x\ge 4$

$2^x\ge 2^2$

$x\ge 2$

$S=[2,+\infty)$

iii) $\log_2(x-1)\ge \log_2(5-x)$

Sol. $x-1\ge 5-x$

$x+x\ge 5+1$

$2x\ge 6$

$x\ge 3$

Check:

$\log_2(x-1)$ and $\log_2(5-x)$ are defined if:

$x-1>0\Rightarrow x>1$ and $5-x>0\Rightarrow -x>-5\Rightarrow x<5$

$S=[3,5)$

iv) $\log_{\frac{1}{3}}(x^2-4x+3)>\log_{\frac{1}{3}}(2x-1)$

Sol. $x^2-4x+3>2x-1$

$x^2-4x-2x+3+1<0$

$x^2-6x+4<0$

$x^2-6x+4=0$

$$x=\frac{6\pm\sqrt{36-16}}{2}=\frac{6\pm\sqrt{20}}{2}=\frac{6\pm\sqrt{4\times5}}{2}$$

$$x=\frac{6\pm2\sqrt{5}}{2}=\frac{2(3\pm\sqrt{5})}{2}=3\pm\sqrt{5}$$

Since $3-\sqrt{5}<x<3+\sqrt{5}$ (because $x^2-6x+4<0$)

Check:

$\log_{\frac{1}{3}}(x^2-4x+3)$ and $\log_{\frac{1}{3}}(2x-1)$

are defined for: $2x-1>0\Rightarrow x>\frac{1}{2}$ and

$x^2-4x+3>0$

$x^2-4x+3=0$

$(x-3)(x-1)=0\Rightarrow x=1,3$

Since $x<1, x>3$ (because $x^2-4x+3>0$)

$S=(3-\sqrt{5},1)\cup(3,3+\sqrt{5})$

v) $\log_2(x-2)+\log_2(x-3)\geq 3$

Sol: $\log_2[(x-2)(x-3)]\geq 3$

$(x-2)(x-3)\geq 2^3$

$x^2-5x+6\geq 8$

$x^2-5x-2\geq 0$

$x^2-5x-2=0$

$$x=\frac{5\pm\sqrt{25+8}}{2}=\frac{5\pm\sqrt{33}}{2}$$

Since $x\leq\frac{5-\sqrt{33}}{2}, x\geq\frac{5+\sqrt{33}}{2}$ (because $x^2-5x-2\geq 0$)

Check:

$\log_2(x-2)$ and $\log_2(x-3)$ are defined if:

$x-2>0\Rightarrow x>2$

$x-3>0\Rightarrow x>3$

$$S=\left[\frac{5+\sqrt{33}}{2},+\infty\right)$$

vi) $\log(x)+\log(x-3)\geq\log(2x)+\log(x-1)$

Sol: $\log[x(x-3)]\geq\log[2x(x-1)]$

$x(x-3)\geq 2x(x-1)$

$x^2-3x\geq 2x^2-2x$

$0\geq 2x^2-2x-x^2+3x$

$0\geq x^2+x$

i.e. $x^2+x\leq 0$

$x^2+x=0$

$x(x+1)=0$

$x=-1,0$

Since $-1\leq x\leq 0$ $\because(x^2+x\leq 0)$

Check:

$\log x, \log(x-3), \log(2x)$, and $\log(x-1)$ are defined for: $x>0$, $x>3$, $x>0$, $x>1$

Solution does not exists $S=\phi$

Exercise 1.4 MULTIPLE CHOICE QUESTIONS

1. Which one of the following satisfy $4^{x-1}=8$

A. $x=\frac{5}{2}$ B. $x=-\frac{5}{2}$ C. $x=8$ D. $x=1$

2. Base condition for the equation $\log_x(x^2-3x+2)=2$

A. $x=\frac{2}{3}$ B. $x\neq 1$ C. both A & B D. None

3. Solve $2^x=16$

A. $x=2$ B. $x=3$ C. $x=4$ D. $x=5$

4. Solve $3^x>27$

A. $x<3$ B. $x>3$ C. $x\geq 8$ D. $x\leq 1$

5. Which of the following is the correct domain of the logarithmic function $\log(x)$?

A. $x\geq 0$ B. $x>0$ C. $x<0$ D. All real numbers.

Answers:

1	A	2	C	3	C	4	B	5	B