Exercise 3.1 — Complete Solution Notes

Mathematics-12 — Unit 3: Integration — Every step shown

Formulas & Identities Used

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$
2. $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx$
3. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f\,dx \pm \int g\,dx$
4. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$
5. $\displaystyle\int e^x\,dx = e^x + c, \quad \int a^x\,dx = \frac{a^x}{\ln a} + c$
6. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
7. $\displaystyle\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$
8. $\displaystyle\int \sin x\,dx = -\cos x + c$
9. $\displaystyle\int \cos x\,dx = \sin x + c$
10. $\displaystyle\int \sec^2 x\,dx = \tan x + c$
11. $\displaystyle\int \csc^2 x\,dx = -\cot x + c$
12. $\displaystyle\int \sin(ax)\,dx = -\frac{\cos ax}{a} + c$
13. $\displaystyle\int \cos(ax)\,dx = \frac{\sin ax}{a} + c$
14. $\sec^2 x = 1 + \tan^2 x, \quad \csc^2 x = 1 + \cot^2 x$
15. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
16. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
17. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Question 1. **Find all the antiderivatives for each derivative.**

Q1 (i) $\dfrac{dy}{dx} = 100x^4$

Integrate both sides with respect to x:

$$\begin{aligned} y &= \int 100x^4\,dx \\ &= 100\int x^4\,dx && \text{(take constant out)} \\ &= 100\cdot\frac{x^{4+1}}{4+1} + c \\ &= 100\cdot\frac{x^5}{5} + c \\ &= 20x^5 + c \end{aligned}$$

$$\boxed{y = 20x^5 + c}$$

Q1 (ii) $\dfrac{dy}{dt} = 2 + 24t^3 - 10t^4$

Integrate both sides with respect to t:

$$\begin{aligned} y &= \int \left(2 + 24t^3 - 10t^4\right) dt \\ &= \int 2\,dt + 24\int t^3\,dt - 10\int t^4\,dt \\ &= 2t + 24 \cdot \frac{t^{3+1}}{3+1} - 10 \cdot \frac{t^{4+1}}{4+1} + c \\ &= 2t + 24 \cdot \frac{t^4}{4} - 10 \cdot \frac{t^5}{5} + c \\ &= 2t + 6t^4 - 2t^5 + c \end{aligned}$$

$$\boxed{y = 2t + 6t^4 - 2t^5 + c}$$

Question 2. **Find each indefinite integral. (Check answer by differentiating.)**

Q2 (i) $\displaystyle\int (1 + 2t)\,dt$

$$\begin{aligned} \int (1 + 2t)\,dt &= \int 1\,dt + 2\int t\,dt \\ &= t + 2 \cdot \frac{t^{1+1}}{1+1} + c \\ &= t + 2 \cdot \frac{t^2}{2} + c \\ &= \boxed{t + t^2 + c} \end{aligned}$$

Check: $\dfrac{d}{dt}\left(t + t^2 + c\right) = 1 + 2t + 0 = 1 + 2t$ ✓

Q2 (ii) $\displaystyle\int (3u^2 + 4u)\,du$

$$\begin{aligned} \int (3u^2 + 4u)\,du &= 3\int u^2\,du + 4\int u\,du \\ &= 3 \cdot \frac{u^3}{3} + 4 \cdot \frac{u^2}{2} + c \\ &= \boxed{u^3 + 2u^2 + c} \end{aligned}$$

Check: $\dfrac{d}{du}\left(u^3 + 2u^2 + c\right) = 3u^2 + 4u$ ✓

Q2 (iii) $\int 100x^8\,dx$

$$\begin{aligned}\int 100x^8\,dx &= 100\int x^8\,dx \\ &= 100\cdot\frac{x^{8+1}}{8+1}+c \\ &= \boxed{\frac{100}{9}x^9+c}\end{aligned}$$

Check: $\dfrac{d}{dx}\left(\dfrac{100}{9}x^9+c\right)=\dfrac{100}{9}\cdot 9x^8=100x^8$ ✓

Q2 (iv) $\int\left(2x^{-2}+3x^{-3}\right)dx$

$$\begin{aligned}\int\left(2x^{-2}+3x^{-3}\right)dx &= 2\int x^{-2}\,dx+3\int x^{-3}\,dx \\ &= 2\cdot\frac{x^{-2+1}}{-2+1}+3\cdot\frac{x^{-3+1}}{-3+1}+c \\ &= 2\cdot\frac{x^{-1}}{-1}+3\cdot\frac{x^{-2}}{-2}+c \\ &= -2x^{-1}-\frac{3}{2}x^{-2}+c \\ &= \boxed{-\frac{2}{x}-\frac{3}{2x^2}+c}\end{aligned}$$

Check: $\dfrac{d}{dx}\left(-2x^{-1}-\dfrac{3}{2}x^{-2}+c\right)=2x^{-2}+3x^{-3}$ ✓

Question 3. Evaluate the following.

Q3 (i) $\int\left(3x+3^x+x^3\right)dx$

$$\begin{aligned}&= 3\int x\,dx+\int 3^x\,dx+\int x^3\,dx \\ &= 3\cdot\frac{x^2}{2}+\frac{3^x}{\ln 3}+\frac{x^4}{4}+c \\ &= \boxed{\frac{3x^2}{2}+\frac{3^x}{\ln 3}+\frac{x^4}{4}+c}\end{aligned}$$

Here $\int a^x dx=\dfrac{a^x}{\ln a}$ was used with $a=3$.

Q3 (ii) $\int x^2\left(x+\frac{1}{x}\right)dx$

First multiply x^2 inside the bracket:

$$x^2\left(x+\frac{1}{x}\right) = x^2\cdot x + x^2\cdot\frac{1}{x} = x^3 + x$$

$$\begin{aligned}\int x^2\left(x+\frac{1}{x}\right)dx &= \int\left(x^3+x\right)dx \\ &= \frac{x^4}{4}+\frac{x^2}{2}+c \\ &\boxed{= \frac{x^4}{4}+\frac{x^2}{2}+c}\end{aligned}$$

Q3 (iii) $\int\left(\sqrt{y}+\frac{1}{y}\right)^2 dy$

Expand using $(a+b)^2 = a^2+2ab+b^2$:

$$\begin{aligned}\left(\sqrt{y}+\frac{1}{y}\right)^2 &= (\sqrt{y})^2 + 2\,(\sqrt{y})\left(\frac{1}{y}\right)+\left(\frac{1}{y}\right)^2 \\ &= y + 2\,y^{1/2}\,y^{-1} + y^{-2} \\ &= y + 2y^{-1/2} + y^{-2}\end{aligned}$$

Now integrate:

$$\begin{aligned}\int\left(y+2y^{-1/2}+y^{-2}\right)dy &= \frac{y^2}{2}+2\cdot\frac{y^{1/2}}{1/2}+\frac{y^{-1}}{-1}+c \\ &= \frac{y^2}{2}+4y^{1/2}-y^{-1}+c \\ &\boxed{= \frac{y^2}{2}+4\sqrt{y}-\frac{1}{y}+c}\end{aligned}$$

Q3 (iv) $\int \frac{\sqrt{x}\,(x+2)}{x}\,dx$

Simplify the integrand first (write $\sqrt{x} = x^{1/2}$):

$$\begin{aligned}\frac{x^{1/2}(x+2)}{x} &= \frac{x^{3/2} + 2x^{1/2}}{x} \\ &= x^{3/2-1} + 2x^{1/2-1} \\ &= x^{1/2} + 2x^{-1/2}\end{aligned}$$

Integrate:

$$\begin{aligned}\int \left(x^{1/2} + 2x^{-1/2}\right) dx &= \frac{x^{3/2}}{3/2} + 2 \cdot \frac{x^{1/2}}{1/2} + c \\ &= \frac{2}{3}x^{3/2} + 4x^{1/2} + c \\ &= \boxed{\frac{2}{3}x^{3/2} + 4\sqrt{x} + c}\end{aligned}$$

Q3 (v) $\int \frac{x+1}{\sqrt{x+2}}\,dx$

Let $u = x + 2$. Then $x = u - 2$, so $x + 1 = u - 1$ and $du = dx$.

$$\begin{aligned}\int \frac{x+1}{\sqrt{x+2}}\,dx &= \int \frac{u-1}{u^{1/2}}\,du \\ &= \int \left(\frac{u}{u^{1/2}} - \frac{1}{u^{1/2}}\right) du \\ &= \int \left(u^{1/2} - u^{-1/2}\right) du \\ &= \frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} + c \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + c\end{aligned}$$

Put $u = x + 2$ back:

$$\boxed{= \frac{2}{3}(x+2)^{3/2} - 2(x+2)^{1/2} + c}$$

Simplified form: take $2(x+2)^{1/2}$ common:

$$= 2(x+2)^{1/2}\left[\frac{x+2}{3} - 1\right] + c = 2\sqrt{x+2} \cdot \frac{x-1}{3} + c = \frac{2}{3}(x-1)\sqrt{x+2} + c$$

Q3 (vi) $\int \frac{(e^x + e^{-x})^3}{e^{3x}}\,dx$

Expand the cube using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = e^x,\ b = e^{-x}$:

$$\begin{aligned}(e^x + e^{-x})^3 &= (e^x)^3 + 3(e^x)^2 e^{-x} + 3e^x (e^{-x})^2 + (e^{-x})^3 \\ &= e^{3x} + 3e^{2x-x} + 3e^{x-2x} + e^{-3x} \\ &= e^{3x} + 3e^x + 3e^{-x} + e^{-3x}\end{aligned}$$

Divide every term by e^{3x}:

$$\begin{aligned}\frac{(e^x + e^{-x})^3}{e^{3x}} &= \frac{e^{3x}}{e^{3x}} + \frac{3e^x}{e^{3x}} + \frac{3e^{-x}}{e^{3x}} + \frac{e^{-3x}}{e^{3x}} \\ &= 1 + 3e^{x-3x} + 3e^{-x-3x} + e^{-3x-3x} \\ &= 1 + 3e^{-2x} + 3e^{-4x} + e^{-6x}\end{aligned}$$

Integrate, using $\int e^{kx}dx = \frac{e^{kx}}{k}$:

$$\int \left(1 + 3e^{-2x} + 3e^{-4x} + e^{-6x}\right) dx = x + 3\cdot\frac{e^{-2x}}{-2} + 3\cdot\frac{e^{-4x}}{-4} + \frac{e^{-6x}}{-6} + c$$

$$\boxed{= x - \frac{3}{2}e^{-2x} - \frac{3}{4}e^{-4x} - \frac{1}{6}e^{-6x} + c}$$

Q3 (vii) $\int \frac{1}{\sqrt{x+1} + \sqrt{x}}\,dx$

Rationalize the denominator (multiply and divide by $\sqrt{x+1} - \sqrt{x}$):

$$\begin{aligned}\frac{1}{\sqrt{x+1} + \sqrt{x}} \times \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} &= \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2} \\ &= \frac{\sqrt{x+1} - \sqrt{x}}{(x+1) - x} \\ &= \frac{\sqrt{x+1} - \sqrt{x}}{1} = (x+1)^{1/2} - x^{1/2}\end{aligned}$$

Integrate:

$$\int \left[(x+1)^{1/2} - x^{1/2}\right] dx = \frac{(x+1)^{3/2}}{3/2} - \frac{x^{3/2}}{3/2} + c$$

$$\boxed{= \frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2} + c}$$

Q3 (viii) $\int 7^{\ln x}\,dx$

Simplify $7^{\ln x}$ first. Write $7 = e^{\ln 7}$:

$$7^{\ln x} = \left(e^{\ln 7}\right)^{\ln x} = e^{(\ln 7)(\ln x)} = \left(e^{\ln x}\right)^{\ln 7} = x^{\ln 7}$$

Now $\ln 7$ is just a constant (not equal to -1), so use the power rule:

$$\int 7^{\ln x}\,dx = \int x^{\ln 7}\,dx$$
$$\boxed{= \frac{x^{\ln 7+1}}{\ln 7 + 1} + c}$$

Q3 (ix) $\int 2^{\log_2 x}\,dx$

Use the rule $a^{\log_a x} = x$:

$$2^{\log_2 x} = x$$
$$\int 2^{\log_2 x}\,dx = \int x\,dx$$
$$\boxed{= \frac{x^2}{2} + c}$$

Q3 (x) $\int \frac{a^x + a^{-x}}{a^x - a^{-x}}\,dx \qquad (a > 0,\ a \neq 1)$

Let $f(x) = a^x - a^{-x}$ (the denominator). Differentiate it:

$$\begin{aligned} f'(x) &= a^x \ln a - a^{-x} \ln a \cdot (-1) \\ &= a^x \ln a + a^{-x} \ln a \\ &= \ln a \left(a^x + a^{-x}\right) \end{aligned}$$

So the numerator is

$$a^x + a^{-x} = \frac{f'(x)}{\ln a}$$

Therefore:

$$\begin{aligned} \int \frac{a^x + a^{-x}}{a^x - a^{-x}}\,dx &= \int \frac{f'(x)/\ln a}{f(x)}\,dx \\ &= \frac{1}{\ln a} \int \frac{f'(x)}{f(x)}\,dx \qquad \text{(Recall Rule 4)} \\ &= \frac{1}{\ln a} \ln |f(x)| + c \\ &\boxed{= \frac{1}{\ln a} \ln \left|a^x - a^{-x}\right| + c} \end{aligned}$$

Q3 (xi) $\displaystyle\int \frac{2^{2x}+1}{2^x}\,dx$

Split into two fractions:

$$\begin{aligned}\frac{2^{2x}+1}{2^x} &= \frac{2^{2x}}{2^x} + \frac{1}{2^x}\\ &= 2^{2x-x} + 2^{-x}\\ &= 2^x + 2^{-x}\end{aligned}$$

Integrate (use $\int a^{kx}dx = \dfrac{a^{kx}}{k\ln a}$):

$$\begin{aligned}\int \left(2^x + 2^{-x}\right) dx &= \frac{2^x}{\ln 2} + \frac{2^{-x}}{(-1)\ln 2} + c\\ &= \frac{2^x}{\ln 2} - \frac{2^{-x}}{\ln 2} + c\\ &\boxed{= \frac{2^x - 2^{-x}}{\ln 2} + c}\end{aligned}$$

Q3 (xii) $\displaystyle\int \frac{x^2+1}{x^3-x}\,dx$

Step 1: Factorize the denominator.

$$x^3 - x = x\left(x^2-1\right) = x(x-1)(x+1)$$

Step 2: Partial fractions.

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Multiply both sides by $x(x-1)(x+1)$:

$$x^2+1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$\begin{aligned}
\text{Put } x=0: &\quad 1 = A(-1)(1) \;\Rightarrow\; A=-1\\
\text{Put } x=1: &\quad 2 = B(1)(2) \;\Rightarrow\; B=1\\
\text{Put } x=-1: &\quad 2 = C(-1)(-2) \;\Rightarrow\; C=1
\end{aligned}$$

Step 3: Integrate.

$$\begin{aligned}
\int \frac{x^2+1}{x^3-x}\,dx &= \int \left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right) dx\\
&= -\ln|x| + \ln|x-1| + \ln|x+1| + c\\
&= \ln\left|\frac{(x-1)(x+1)}{x}\right| + c\\
&\boxed{= \ln\left|\frac{x^2-1}{x}\right| + c}
\end{aligned}$$

Check: $\dfrac{d}{dx}\left[\ln|x^2-1| - \ln|x|\right] = \dfrac{2x}{x^2-1} - \dfrac{1}{x} = \dfrac{2x^2-(x^2-1)}{x(x^2-1)} = \dfrac{x^2+1}{x^3-x}$ ✓

Question 4. Evaluate the following.

Q4 (i) $\int \csc x\,dx$

Multiply and divide by $(\csc x - \cot x)$:

$$\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx$$
$$= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x}\,dx$$

Let $f(x) = \csc x - \cot x$. Then

$$f'(x) = -\csc x \cot x + \csc^2 x = \csc^2 x - \csc x \cot x$$

which is exactly the numerator. So the integral is of the form $\int \frac{f'(x)}{f(x)}dx$:

$$\boxed{\int \csc x\,dx = \ln|\csc x - \cot x| + c}$$

Q4 (ii) $\int \cot^2 x\,dx$

Use $\csc^2 x = 1 + \cot^2 x \;\Rightarrow\; \cot^2 x = \csc^2 x - 1$:

$$\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx$$
$$= \int \csc^2 x\,dx - \int 1\,dx$$
$$\boxed{= -\cot x - x + c}$$

Q4 (iii) $\int \sec^4 x\,dx$

Split $\sec^4 x = \sec^2 x \cdot \sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ for one of them:

$$\int \sec^4 x\,dx = \int \sec^2 x \cdot \sec^2 x\,dx$$
$$= \int (1 + \tan^2 x)\sec^2 x\,dx$$

Let $u = \tan x$. Then $du = \sec^2 x\,dx$:

$$= \int (1 + u^2)\,du$$
$$= u + \frac{u^3}{3} + c$$
$$\boxed{= \tan x + \frac{\tan^3 x}{3} + c}$$

Q4 (iv) $\int \sin x \cdot \cos 7x \, dx$

Use $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ with $A = x,\ B = 7x$:

$$\begin{aligned} 2\sin x \cos 7x &= \sin(x+7x) + \sin(x-7x) \\ &= \sin 8x + \sin(-6x) \\ &= \sin 8x - \sin 6x \\ \Rightarrow\ \sin x \cos 7x &= \frac{1}{2}(\sin 8x - \sin 6x) \end{aligned}$$

Integrate:

$$\begin{aligned} \int \sin x \cos 7x \, dx &= \frac{1}{2}\left[\int \sin 8x \, dx - \int \sin 6x \, dx\right] \\ &= \frac{1}{2}\left[-\frac{\cos 8x}{8} + \frac{\cos 6x}{6}\right] + c \\ &\boxed{= -\frac{\cos 8x}{16} + \frac{\cos 6x}{12} + c} \end{aligned}$$

Q4 (v) $\int \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x}\, dx$

Use $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ with $a = \cos x,\ b = \sin x$:

$$\begin{aligned} \frac{\cos^3 x - \sin^3 x}{\cos x - \sin x} &= \frac{(\cos x - \sin x)\left(\cos^2 x + \cos x \sin x + \sin^2 x\right)}{\cos x - \sin x} \\ &= \cos^2 x + \sin^2 x + \sin x \cos x \\ &= 1 + \sin x \cos x \\ &= 1 + \frac{1}{2}\sin 2x \qquad (\text{since } \sin 2x = 2\sin x \cos x) \end{aligned}$$

Integrate:

$$\begin{aligned} \int \left(1 + \frac{1}{2}\sin 2x\right) dx &= x + \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c \\ &\boxed{= x - \frac{1}{4}\cos 2x + c} \end{aligned}$$

Q4 (vi) $\int \frac{\cos 2x}{\sin x + \cos x}\,dx$

Use $\cos 2x = \cos^2 x - \sin^2 x$ and factorize (difference of squares):

$$\begin{aligned}\frac{\cos 2x}{\sin x + \cos x} &= \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x} \\ &= \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sin x + \cos x} \\ &= \cos x - \sin x\end{aligned}$$

Integrate:

$$\int (\cos x - \sin x)\,dx = \sin x - (-\cos x) + c$$

$$\boxed{= \sin x + \cos x + c}$$

Q4 (vii) $\int \frac{1}{1 + \cos x}\,dx$

Use $\cos x = 2\cos^2 \frac{x}{2} - 1 \;\Rightarrow\; 1 + \cos x = 2\cos^2 \frac{x}{2}$:

$$\begin{aligned}\int \frac{1}{1+\cos x}\,dx &= \int \frac{1}{2\cos^2 \frac{x}{2}}\,dx \\ &= \frac{1}{2}\int \sec^2 \frac{x}{2}\,dx \\ &= \frac{1}{2}\cdot \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c \qquad \left(\int \sec^2(ax)\,dx = \frac{\tan ax}{a}\right)\end{aligned}$$

$$\boxed{= \tan \frac{x}{2} + c}$$

Q4 (viii) $\int \frac{1 - \cos 2x}{1 + \cos 2x}\,dx$

Use $1 - \cos 2x = 2\sin^2 x$ and $1 + \cos 2x = 2\cos^2 x$:

$$\begin{aligned}\frac{1 - \cos 2x}{1 + \cos 2x} &= \frac{2\sin^2 x}{2\cos^2 x} \\ &= \tan^2 x \\ &= \sec^2 x - 1\end{aligned}$$

Integrate:

$$\int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1\,dx$$

$$\boxed{= \tan x - x + c}$$

Q4 (ix) $\displaystyle\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta}\, dx$

Use $\cos 2A = 2\cos^2 A - 1$ in the numerator:

$$\begin{aligned}\cos 2x - \cos 2\theta &= \left(2\cos^2 x - 1\right) - \left(2\cos^2 \theta - 1\right) \\ &= 2\cos^2 x - 2\cos^2 \theta \\ &= 2\left(\cos^2 x - \cos^2 \theta\right) \\ &= 2(\cos x - \cos\theta)(\cos x + \cos\theta)\end{aligned}$$

Divide by the denominator:

$$\begin{aligned}\frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} &= \frac{2(\cos x - \cos\theta)(\cos x + \cos\theta)}{\cos x - \cos\theta} \\ &= 2(\cos x + \cos\theta)\end{aligned}$$

Integrate (here θ is a constant, so $\cos\theta$ is a constant):

$$\begin{aligned}\int 2(\cos x + \cos\theta)\, dx &= 2\int \cos x\, dx + 2\cos\theta \int 1\, dx \\ &\boxed{= 2\sin x + 2x\cos\theta + c}\end{aligned}$$

— End of Exercise 3.1 —

Prepared by The Genius Science Academy, Lahore